1 The Derivative of ln(x)

The first term or semester of most calculus courses will include the limit definition of the derivative and will work out, long hand, a number of specific examples to illustrate more general rules for quickly developing derivatives. For polynomials, these rules are relatively easily developed and applied by students. Even in the case of certain transcendental functions, such as sine and cosine where the infinite series can be manipulated and its results then compared, students can be quickly shown why it is that:

$$\frac{d}{dx}\sin x = \cos x \tag{1}$$

A simple inspection of the each series and taking the derivative, term by term, is often sufficient to get the point across without too much difficulty. Of course, that demonstration assumes that the infinite series is granted by the student, to start with. And although Taylor's can be brought in, explained in principle, and used to develop the series, the straight-forward application of Taylor's assumes the derivatives are known beforehand so that the constants of consecutive terms in the Taylor's expansion can be computed.

Luckily, all that can be avoided. Trigonometric identities and a little imaginative use of conjugates can be applied to the limit definition for the derivative to make the case without resorting to the infinite series. In other words, a student's earlier trigonometric foundations are sufficient to allow them to follow a clear argument in this case. So, if pressed, a teacher can lead students through the details using their prior training and fresh knowledge on developing results for limit problems. However, this isn't so easy in the case of showing:

$$\frac{d}{dx}\ln x = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x} \tag{2}$$

A student may be left with accepting the declaration without seeming to have any way of confirming the result. But that is appearances. The demonstration can be made using knowledge they probably already have prior to taking the first term or semester of calculus together with the early foundations being laid in that class. But to get there we need to consider an interesting limit problem.

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2 Two Curious Questions

There is a fascinating limit problem that evolves from a very practical question — namely, effective interest rates — as well as a curious probability question.

2.1 Interest rates

Interest can be compounded — meaning that not only is interest paid on the principle amount but also that interest is also paid on earlier accumulated interest, as well. For example, a 10% per annum might be specified, but interest might be accrued and then considered a part of the principle for further calculations, more often than once a year. Let's say the number of equal periods used in a year is n. To compute the effective interest rate over the period, the annual interest rate is equally divided and then applied, yielding the following equation:

$$r_e = \left(1+\frac{r_a}{n}\right)^n - 1, \quad \textit{where } r_e \textit{ is the effective rate and } r_a \textit{ is the simple rate} \tag{3}$$

But what if we compound this constantly? In other words, what happens as $n \to \infty$?

2.2 Probability

My son asked himself a simple question. What if the odds of something happening were fairly unlikely, but we expanded the number of trials to compensate for that? In other words, what if we increased the number of trials in inverse proportion to the odds? What chance would there still be for a failure, then? Perform 6 trials when there is only 1 chance in 6? Or perform 100 trials when there is only 1 chance in 100? Etc. He wanted to know what the odds for not one success might be, if more trials were performed for less likely events.

He was able to express the basic idea from his knowledge of combining probabilities of independent events. The probability, between 0 and 1, is:

$$P_{failure} = \left(1-\frac{1}{x}\right)^x, \quad \textit{given each independent event succeeds once in } x \textit{ trials} \tag{4}$$

The question he then asked was, what happens as the odds decline arbitrarily and the number of trials increase, proportionately, as $x \to \infty$?

Both of these questions can be answered by exploring this limit:

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n, \quad \textit{where } a \textit{ is some constant we can choose} \tag{5}$$

This is where we will now focus our attention.

3 Two Curious Questions Answered

Let's take a few initial steps along the way and then focus on a familiar portion:

$$\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^x\right] = \lim_{x\to\infty}\left[\left(\frac{x+a}{x}\right)^x\right] = \lim_{x\to\infty}\left[\frac{(x+a)^x}{x^x}\right] \tag{6}$$

Expanding the numerator portion of the right side in (6) provides some idea about how we might express this in a slightly different form: the first five expansions, starting with $x=0$ and proceeding to $x=5$. I'll leave **x** in the expanded terms for familiarity and placeholder purposes, but keep in mind that it has a different numeric value in each case:

$$\begin{array}{rcccc} (x+a)^0 & = & 1 & & x = 0 \\ (x+a)^1 & = & x+a & & x = 1 \\ (x+a)^2 & = & x^2+2\cdot a\cdot x+a^2 & & x = 2 \\ (x+a)^3 & = & x^3+3\cdot a\cdot x^2+3\cdot a^2\cdot x+a^3 & & x = 3 \\ (x+a)^4 & = & x^4+4\cdot a\cdot x^3+6\cdot a^2\cdot x^2+4\cdot a^3\cdot x+a^4 & & x = 4 \\ (x+a)^5 & = & x^5+5\cdot a\cdot x^4+10\cdot a^2\cdot x^3+10\cdot a^3\cdot x^2+5\cdot a^4\cdot x+a^5 & & x = 5 \end{array} \tag{7}$$

This is a familiar binomial series expansion. The constants in each term come from Pascal's triangle and the general form of the binomial series identity is:

$$(x+a)^n = \sum_{k=0}^{n}\left[\binom{n}{k}\cdot x^{n-k}\cdot a^k\right],\ \textit{where the binomial coefficient}\binom{n}{k} = \frac{n!}{(x-k)!\cdot k!} \tag{8}$$

We can then substitute the summation in (8) into the right-side expression in (6) to get:

$$\lim_{x\to\infty}\left[\frac{(x+a)^x}{x^x}\right] = \lim_{x\to\infty}\frac{\sum_{k=0}^{x}\left[\frac{x!}{(x-k)!\cdot k!}\cdot x^{x-k}\cdot a^k\right]}{x^x} = \lim_{x\to\infty}\frac{1}{x^x}\cdot\sum_{k=0}^{x}\left[\frac{x!}{(x-k)!\cdot k!}\cdot x^{x-k}\cdot a^k\right] \tag{9}$$

The term $\frac{1}{x^x}$ doesn't vary on k, so we can move it into the summation with the plan of then combining it with the term, x^{x-k}, found inside the summation expression:

$$\cdots = \lim_{x\to\infty}\sum_{k=0}^{x}\left[\frac{x!}{(x-k)!\cdot k!}\cdot\left(\frac{x^{x-k}}{x^x}\right)\cdot a^k\right] = \lim_{x\to\infty}\sum_{k=0}^{x}\left[\frac{x!}{(x-k)!\cdot k!}\cdot x^{-k}\cdot a^k\right] \tag{10}$$

I'm going to re-arrange the terms in (10) a little bit, so that we can focus on an interesting part of the expression:

$$\cdots = \lim_{x\to\infty}\sum_{k=0}^{x}\left[\left(\frac{x!}{(x-k)!\cdot x^k}\right)\cdot\frac{a^k}{k!}\right] \tag{11}$$

Let's look at the first factor in the summation term, considering different values of k.

$$\lim_{x\to\infty}\frac{x!}{(x-0)!\cdot x^0} = \lim_{x\to\infty}\frac{x!}{x!\cdot 1} = \lim_{x\to\infty}\frac{x!}{x!} = 1, \quad where\ k=0$$
$$\lim_{x\to\infty}\frac{x!}{(x-1)!\cdot x} = \lim_{x\to\infty}\frac{x!}{x!} = 1, \quad where\ k=1$$
$$\lim_{x\to\infty}\frac{x!}{(x-2)!\cdot x\cdot x} = \lim_{x\to\infty}\frac{x!}{x!\cdot\frac{x}{x-1}} = \lim_{x\to\infty}\frac{x-1}{x} = 1, \quad where\ k=2 \tag{12}$$

$$\lim_{x\to\infty}\prod_{j=0}^{k-1}\frac{x-j}{x} = 1, \qquad k > 0$$

As $x\to\infty,$ the term remains 1 for any finite k. To summarize:

$$\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^x\right] = \lim_{x\to\infty}\sum_{k=0}^{x}\left[\left(\frac{x!}{(x-k)!\cdot x^k}\right)\cdot\frac{a^k}{k!}\right] = \lim_{x\to\infty}\sum_{k=0}^{x}\left[\frac{a^k}{k!}\right] = \sum_{k=0}^{\infty}\left[\frac{a^k}{k!}\right] = e^a \tag{13}$$

The infinite series indicated in the latter part of (13) is a well-recognized expansion.

We can now answer the earlier two questions.

3.1 Interest rates

The effective interest rate, compounded constantly, can now be completed as:

$$r_e = \lim_{n\to\infty}\left[\left(1+\frac{r_a}{n}\right)^n\right]-1 = e^{r_a}-1 \tag{14}$$

An interest rate of 8% per annum, compounded constantly, becomes an effective rate of about 8.33%. In other words, \$100 would yield \$8.33 in one year, rather than just the \$8 that would have otherwise happened if simple interest were being applied, rather than being compounded constantly.

3.2 Probability

A similar answer applies to the probability question, as the event likelihood diminishes but also the number of trials increases. The result looks like:

$$P_{failure} = \lim_{x\to\infty}\left[\left(1-\frac{1}{x}\right)^x\right] = \lim_{x\to\infty}\left[\left(1+\frac{-1}{x}\right)^x\right] = e^{-1} \approx 0.36788 \tag{15}$$

Now on towards solving the derivative of ln(x).

4 Returning to the Derivative of ln(x)

A really interesting question comes from wondering how it might be that we can find the derivative of a transcendental equation to be a simple rational fraction! How is it that:

$$d \ln x \;=\; \frac{dx}{x} \tag{16}$$

Expressed slightly differently and using the limit theorem for finding the derivative, we have:

$$\frac{d}{dx}\ln x \;=\; \lim_{h\to 0}\frac{\ln(x+h) \;-\; \ln(x)}{h} \;=\; \frac{1}{x} \tag{17}$$

From our knowledge about subtraction of logarithms, we can rephrase this as:

$$\frac{d}{dx}\ln x \;=\; \lim_{h\to 0}\frac{\ln(x+h) \;-\; \ln(x)}{h} \;=\; \lim_{h\to 0}\left[\frac{1}{h}\cdot\ln\left(\frac{x+h}{x}\right)\right] \;=\; \lim_{h\to 0}\left[\frac{1}{h}\cdot\ln\left(1+\frac{h}{x}\right)\right] \tag{18}$$

Of course, that doesn't help a lot. But the solution is near to hand. Let's rephrase (18) this way:

$$\cdots \;=\; \lim_{h\to 0}\left[\frac{1}{x}\cdot\frac{x}{h}\cdot\ln\left(1+\frac{h}{x}\right)\right] \tag{19}$$

The term, $\frac{1}{x}$, can be moved outside the limit since it doesn't vary as $h\to 0$.

$$\cdots \;=\; \frac{1}{x}\cdot\lim_{h\to 0}\left[\frac{x}{h}\cdot\ln\left(1+\frac{h}{x}\right)\right] \tag{20}$$

We can now proceed to apply another property when multiplying a logarithm by a term:

$$\cdots \;=\; \frac{1}{x}\cdot\lim_{h\to 0}\left[\ln\left\{\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right\}\right] \tag{21}$$

Given our earlier work, we can almost see the answer. Let's create a new variable, defined as $n=\frac{x}{h}$ and apply it to our formula. Keep in mind that as $h\to 0$, $n\to\infty$.

$$\cdots \;=\; \frac{1}{x}\cdot\lim_{n\to\infty}\left[\ln\left\{\left(1+\frac{1}{n}\right)^{n}\right\}\right] \tag{22}$$

Of course, we now know what happens to that expression inside the logarithm! It becomes the value of e. It is easy to see where this is headed, now:

$$\frac{d}{dx}\ln x \;=\; \frac{1}{x}\cdot\lim_{n\to\infty}\left[\ln\left\{\left(1+\frac{1}{n}\right)^{n}\right\}\right] \;=\; \frac{1}{x}\cdot\lim_{n\to\infty}\left[\ln(e)\right] \;=\; \frac{1}{x}\cdot\lim_{n\to\infty} 1 \;=\; \frac{1}{x} \tag{23}$$

5 Summary

Both the fun, perhaps as well as one of those necessary facts of doing a little mathematics, is that sometimes an excursion into seemingly unrelated areas may provide a tool needed at a different time and where, without having had that earlier exposure, the approach might not have been so easily recognized.

The trip might be stimulated by a practical question, such as the one about interest rates, or else by way of a hypothetical probability question born more out of modest curiosity. Or something else, still. But searching and finding that answer provided a useful tool to help find the derivative for an interesting transcendental function.

Serendipity is a familiar face in mathematics.